微分積分Ⅰ　小課題第 19 回

裏面にある略解をもとに丸付けをすること。裏面も解答に使ってもよいです。授業の質問も書いてくれれば回答します。名前等、忘れずにていねいに書いてください！

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ ← よく出る！

2年 ___ 科 ___ 番　氏名 ________________

1. 次の定積分を計算せよ。

(1) $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$x = \sin\theta \ (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

$dx = \cos\theta\, d\theta$

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cos\theta$

x	$-1 \longrightarrow 1$
θ	$-\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot \cos\theta\, d\theta$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= \left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= \frac{\pi}{2}$ 」

(2) $\int_{-1}^{\sqrt{2}} \frac{1}{\sqrt{4-x^2}}\,dx$

$x = 2\sin\theta \ (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

$dx = 2\cos\theta\, d\theta$

$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

x	$-1 \longrightarrow \sqrt{2}$
θ	$-\frac{\pi}{6} \longrightarrow \frac{\pi}{4}$

↑

$2\cos\theta = -1 \ (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

$\cos\theta = -\frac{1}{2}$

$\therefore \theta = -\frac{\pi}{6}$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2\cos\theta} \cdot 2\cos\theta\, d\theta$

$= \int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta$

$= [\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \frac{\pi}{4} + \frac{\pi}{6}$

$= \frac{5}{12}\pi$ 」

(3) $\int_{-\frac{3}{2}}^{\sqrt{3}} \sqrt{3-x^2}\,dx$

$x = \sqrt{3}\sin\theta \ (-\frac{\pi}{2} \le \theta \le \frac{\pi}{2})$

$dx = \sqrt{3}\cos\theta\, d\theta$

$\sqrt{3-x^2} = \sqrt{3-3\sin^2\theta} = \sqrt{3}\cos\theta$

x	$-\frac{3}{2} \rightarrow \sqrt{3}$
θ	$-\frac{\pi}{3} \longrightarrow \frac{\pi}{2}$

$= \int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \sqrt{3}\cos\theta \cdot \sqrt{3}\cos\theta\, d\theta$

$= 3\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} \left(\frac{1}{2} + \frac{1}{2}\cos 2\theta\right) d\theta$

$= 3\left[\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right]_{-\frac{\pi}{3}}^{\frac{\pi}{2}}$

$= \frac{3}{2}\left(\frac{\pi}{2} + \frac{\pi}{3}\right) + \frac{3}{4}\left(\sin\pi - \sin\left(-\frac{2}{3}\pi\right)\right)$

$= \frac{5}{4}\pi + \frac{3\sqrt{3}}{8}$ 」

(4) $\int_{0}^{\frac{2\sqrt{3}}{3}} \frac{1}{4+x^2}\,dx$

← $\frac{1}{a}\int \frac{1}{1+x^2}dx$ の形にする！

$= \int_{0}^{\frac{2\sqrt{3}}{3}} \frac{1}{4\left(1+\frac{x^2}{4}\right)}dx$

$= \frac{1}{4}\int_{0}^{\frac{2\sqrt{3}}{3}} \frac{1}{1+\left(\frac{x}{2}\right)^2}dx$

$$\boxed{\int \frac{1}{1+\left(\frac{x}{a}\right)^2}dx = \left(\mathrm{Arctan}\frac{x}{a}\right) \times a}$$

$= \frac{1}{4}\left[\mathrm{Arctan}\frac{x}{2} \times 2\right]_{0}^{\frac{2\sqrt{3}}{3}}$

$= \frac{1}{2}\left(\mathrm{Arctan}\frac{\sqrt{3}}{3} - \mathrm{Arctan}\,0\right)$

$= \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{12}$ 」

$x = \mathrm{Arctan}\frac{\sqrt{3}}{3}$

$\Leftrightarrow \tan x = \frac{\sqrt{3}}{3}$

$\Leftrightarrow x = \frac{\pi}{6}$

(5) $\int_{-\sqrt{2}}^{\sqrt{6}} \frac{1}{2+x^2}\,dx$

$= \int_{-\sqrt{2}}^{\sqrt{6}} \frac{1}{2} \cdot \frac{1}{1+\left(\frac{x}{\sqrt{2}}\right)^2}dx$

$= \frac{1}{2}\left[\mathrm{Arctan}\frac{x}{\sqrt{2}} \times \sqrt{2}\right]_{-\sqrt{2}}^{\sqrt{6}}$

$= \frac{\sqrt{2}}{2}\left(\mathrm{Arctan}\sqrt{3} - \mathrm{Arctan}(-1)\right)$

$= \frac{\sqrt{2}}{2}\left(\frac{\pi}{3} - \left(-\frac{\pi}{4}\right)\right)$

$= \frac{7\sqrt{2}}{24}\pi$ 」

部分積分の形 →

(6) $\int_{1}^{2e} x\log x\,dx$

$= \int_{1}^{2e} \left(\frac{1}{2}x^2\right)' \log x\, dx$

$= \left[\frac{1}{2}x^2 \cdot \log x\right]_{1}^{2e} - \int_{1}^{2e} \frac{1}{2}x^2 \cdot (\log x)'\, dx$

$= \frac{1}{2} \cdot (4e^2) \cdot \log 2e - \frac{1}{2} \cdot 1 \cdot \log 1$

$- \int_{1}^{2e} \frac{1}{2}x\, dx$

$= 2e^2(\log 2 + \log e) - \frac{1}{2}\left[\frac{1}{2}x^2\right]_{1}^{2e}$

$= 2e^2\log 2 + 2e^2 - \frac{1}{4}(4e^2 - 1)$

今日は裏にも問題があります！

$= 2e^2\log 2 + e^2 + \frac{1}{4}$

$$(e^{2x})' = 2e^{2x}$$

(7) $\displaystyle\int_0^1 xe^{2x}\,dx$

$$= \int_0^1 x \cdot \left(\tfrac{1}{2}e^{2x}\right)' dx$$

$$= \left[x \cdot \tfrac{1}{2}e^{2x}\right]_0^1 - \int_0^1 1 \cdot \tfrac{1}{2}e^{2x}\,dx$$

$$= \tfrac{1}{2}e^2 - \tfrac{1}{2}\left[\tfrac{1}{2}e^{2x}\right]_0^1$$

$$= \tfrac{1}{2}e^2 - \tfrac{1}{4}(e^2 - 1)$$

$$= \tfrac{1}{4}e^2 + \tfrac{1}{4}$$

(8) $\displaystyle\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} x\sin 4x\,dx$

$$= \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} x\left(-\tfrac{1}{4}\cos 4x\right)' dx$$

$$= \left[x \cdot \left(-\tfrac{1}{4}\cos 4x\right)\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} - \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 1 \cdot \left(-\tfrac{1}{4}\cos 4x\right) dx$$

$$= -\tfrac{1}{4}\left(\tfrac{\pi}{4}\cos\pi - \tfrac{\pi}{8}\cos\tfrac{\pi}{2}\right) + \tfrac{1}{4}\left[\tfrac{1}{4}\sin 4x\right]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= \tfrac{\pi}{16} + \tfrac{1}{16}\left(\sin\pi - \sin\tfrac{\pi}{2}\right)$$

$$= \tfrac{\pi}{16} - \tfrac{1}{16}$$

$$I_1 = \int_0^1 x^m \cdot (1-x)\,dx$$

$$= \int_0^1 (x^m - x^{m+1})\,dx$$

$$= \left[\tfrac{1}{m+1}x^{m+1} - \tfrac{1}{m+2}x^{m+2}\right]_0^1$$

$$= \tfrac{1}{m+1} - \tfrac{1}{m+2}$$

$$= \tfrac{1}{(m+2)(m+1)}$$

2. $I_n = \displaystyle\int_0^1 x^m(1-x)^n\,dx$ (m, n は自然数) とおくとき、次の問いに答えよ。

(1) 部分積分法を適用して、

$$I_n = \frac{n}{m+n+1} I_{n-1}\ (n \geq 2)$$

が成り立つことを示せ。

$$I_n = \int_0^1 \left(\tfrac{1}{m+1}x^{m+1}\right)'(1-x)^n\,dx$$

$$= \left[\tfrac{1}{m+1}x^{m+1} \cdot (1-x)^n\right]_0^1 - \int_0^1 \tfrac{1}{m+1}x^{m+1} \cdot n(1-x)^{n-1} \cdot (-1)\,dx$$

$$= \tfrac{n}{m+1}\int_0^1 x \cdot x^m (1-x)^{n-1}\,dx$$

$$= \tfrac{n}{m+1}\int_0^1 \{1-(1-x)\} \cdot x^m(1-x)^{n-1}\,dx$$

$$= \tfrac{n}{m+1}\int_0^1 x^m(1-x)^{n-1}\,dx - \tfrac{n}{m+1}\int_0^1 x^m(1-x)^n\,dx$$

$$I_n = \tfrac{n}{m+1}I_{n-1} - \tfrac{n}{m+1}I_n$$

$$I_n + \tfrac{n}{m+1}I_n = \tfrac{n}{m+1}I_{n-1}$$

$$\tfrac{m+n+1}{m+1}I_n = \tfrac{n}{m+1}I_{n-1}$$

$$I_n = \tfrac{n}{m+n+1}I_{n-1}$$

(2) $I_n = \dfrac{m!\,n!}{(m+n+1)!}$ となることを示せ。=

(1)より、$I_n = \dfrac{n}{m+n+1}I_{n-1}$

$$= \tfrac{n}{m+n+1} \cdot \tfrac{n-1}{m+n}I_{n-2}$$

$$= \tfrac{n}{m+n+1} \cdot \tfrac{n-1}{m+n} \cdot \tfrac{n-2}{m+n-1}I_{n-3}$$

$$= \cdots$$

$$= \tfrac{n}{m+n+1} \cdot \tfrac{n-1}{m+n} \cdot \tfrac{n-2}{m+n-1} \cdots\cdots \tfrac{3}{m+4} \cdot \tfrac{2}{m+3}I_1$$

$$= \tfrac{m!\,n!}{(m+n+1)!}$$

略解

1. (1) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = \frac{\pi}{2}$ $\quad x = \sin\theta$, $dx = \cos\theta\,dx$

(2) $\displaystyle\int_{-\frac{\pi}{6}}^{\frac{\pi}{4}} d\theta = \frac{5}{12}\pi$ $\quad x = 2\sin\theta$, $dx = 2\cos\theta\,dx$

(3) $\displaystyle\int_{-\frac{\pi}{3}}^{\frac{\pi}{2}} 3\cos^2\theta\,d\theta = \frac{10\pi + 3\sqrt{3}}{8}$ $\quad x = \sqrt{3}\sin\theta$, $dx = \sqrt{3}\cos\theta\,dx$

(4) $\displaystyle\frac{1}{4}\left[2\,\mathrm{Arctan}\,\frac{x}{2}\right]_0^{\frac{2\sqrt{3}}{3}} = \frac{\pi}{12}$

(5) $\displaystyle\frac{1}{2}\left[\sqrt{2}\,\mathrm{Arctan}\,\frac{x}{\sqrt{2}}\right]_{-\sqrt{2}}^{\sqrt{6}} = \frac{7\sqrt{2}\pi}{24}$

(6) $\displaystyle\int_1^{2e} \left(\frac{1}{2}x^2\right)'\log x\,dx = e^2 + 2e^2\log 2 + \frac{1}{4}$

(7) $\displaystyle\int_0^1 x\left(\frac{1}{2}e^{2x}\right)'dx = \frac{1}{4}e^2 + \frac{1}{4}$

(8) $\displaystyle\frac{\pi - 1}{16}$

2. (1) $I_n = \displaystyle\int_0^1 \left(\frac{1}{m+1}x^{m+1}\right)'(1-x)^n\,dx$ (2) $I_n = \dfrac{n}{m+n+1}I_{n-1} = \dfrac{n}{m+n+1} \cdot \dfrac{n-1}{m+n}I_{n-2} = \ldots$